Chapter 2

Examining Powers and Roots

In This Chapter

- Working through operations with exponents
- Eliminating the negativity of negative exponents
- Getting to the root of roots

Exponents were developed so that mathematicians wouldn't have to keep repeating themselves! What is an exponent? An *exponent* is the small, superscripted number to the upper right of the larger number that tells you how many times you multiply the larger number, called the *base*.

Expanding and Contracting with Exponents

When algebra was first written with symbols — instead of with all words — there were no exponents. If you wanted to multiply the variable *y* times itself six times, you'd write it: *yyyyyy*. Writing the variable over and over can get tiresome, so the wonderful system of exponents was developed.

The base of an exponential expression can be any real number (see Chapter 1 for more on real numbers). The exponent can be any real number, too, as long as rules involving radicals aren't violated (see "Circling around Square Roots," later in this chapter). An exponent can be positive, negative, fractional, or even a radical. What power!

When a number x is involved in repeated multiplication of x times itself, then the number n can be used to describe how many multiplications are involved: $x^n = x \cdot x \cdot x \cdot x \cdot x \ldots n$ times.

Even though the x in the expression x^n can be any real number and the n can be any real number, they can't both be 0 at the same time. For example, 0^0 really has no meaning in algebra. It takes a calculus course to prove why this restriction is so. Also, if x is equal to 0, then n can't be negative.

Here are two examples using exponential notation:

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \frac{1}{5^3} = \frac{1}{5 \cdot 5 \cdot 5} = \frac{1}{125}$

When the exponent is negative, you apply the rule involving rewriting negative exponents before writing the product. (See "Taking on the Negativity of Exponents," later in this chapter.)

Exhibiting Exponent Products

You can multiply many exponential expressions together without having to change their form into the big or small numbers they represent. The only requirement is that the bases of the exponential expressions that you're multiplying have to be the same. The answer is then a nice, neat exponential expression.

You *can* multiply $2^4 \cdot 2^6$ and $a^5 \cdot a^8$, but you *cannot* multiply $3^6 \cdot 4^7$ using the rule, because the bases are not the same.

To multiply powers of the same base, add the exponents together: $x^a \cdot x^b = x^{a+b}$.

Here are two examples of finding the products of the numbers by adding the exponents:

- $2^4 \cdot 2^6 = 2^{4+6} = 2^{10}$
- $a^5 \cdot a^8 = a^{13}$

Often, you find algebraic expressions with a whole string of factors; you want to simplify the expression, if possible. When there's more than one base in an expression with powers, you combine the numbers with the same bases, find the values, and then write them all together.

Here's how to simplify the following expressions:

- $\mathbf{3^2 \cdot 2^2 \cdot 3^3 \cdot 2^4 = 3^{2+3} \cdot 2^{2+4} = 3^5 \cdot 2^6}$**:** The two factors with base 3 combine, as do the two factors with base 2.
- $\mathbf{4x^6y^5x^4y = 4x^{6+4}y^{5+1} = 4x^{10}y^6}$**:** The number 4 is a coefficient, which is written before the rest of the factors.

When there's no exponent showing, such as with y, you assume that the exponent is 1. In the preceding example, you can see that the factor y was written as y^1 so its exponent could be added to that in the other y factor.

Taking Division to Exponents

You can divide exponential expressions, leaving the answers as exponential expressions, as long as the bases are the same. Division is the opposite of multiplication, so it makes sense that, because you add exponents when multiplying numbers with the same base, you *subtract* the exponents when dividing numbers with the same base. Easy enough?

To divide powers with the same base, subtract the exponents: $\frac{x^a}{x^b} = x^a \div x^b = x^{a-b}$, where x can be any real number except 0. (***Remember:*** You can't divide by 0.)

Here are two examples of simplifying expressions by dividing:

- $\mathbf{2^{10} \div 2^4 = 2^{10-4} = 2^6}$**:** These exponentials represent the equation $1{,}024 \div 16 = 64$. It's much easier to leave the numbers as bases with exponents.
- $\mathbf{\frac{4x^6y^3z^2}{2x^4y^3z} = 2x^{6-4}y^{3-3}z^{2-1} = 2x^2y^0z^1 = 2x^2z}$

Did you wonder where the y factor went? For more on y^0, read on.

Taking on the Power of Zero

If x^3 means $x \cdot x \cdot x$, what does x^0 mean? Well, it doesn't mean x times 0, so the answer isn't 0. x represents some unknown real number; real numbers can be raised to the 0 power — except that the base just can't be 0. To understand how this works, use the following rule for division of exponential expressions involving 0.

Any number to the power of 0 equals 1 as long as the base number is not 0. In other words, $a^0 = 1$ as long as $a \neq 0$.

Here are two examples of simplifying, using the rule that when you raise a real number a to the 0 power, you get 1:

- $m^2 \div m^2 = m^{2-2} = m^0 = 1$.
- $4x^3y^4z^7 \div 2x^3y^3z^7 = 2x^{3-3}y^{4-3}z^{7-7} = 2x^0y^1z^0 = 2y$. Both x and z end up with exponents of 0, so those factors become 1. Neither x nor z may be equal to 0.

Taking on the Negativity of Exponents

Negative exponents are a neat little creation. They mean something very specific and have to be handled with care, but they are oh, so convenient to have.

You can use a negative exponent to write a fraction without writing a fraction! Using negative exponents is a way to combine expressions with the same base, whether the different factors are in the numerator or denominator. It's a way to change division problems into multiplication problems.

Negative exponents are a way of writing powers of fractions or decimals without using the fraction or decimal. For example, instead of writing $\left(\frac{1}{10}\right)^{14}$, you can write 10^{-14}.

The reciprocal of x^a is $\frac{1}{x^a}$, which can be written as x^{-a}. The variable x is any real number except 0, and a is any real number. Also, to get rid of the negative exponent, you write: $x^{-a} = \frac{1}{x^a}$.

Here are two examples of changing numbers with negative exponents to fractions with positive exponents:

- $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$. The reciprocal of 2^3 is $\frac{1}{2^3} = 2^{-3}$.
- $6^{-1} = \frac{1}{6}$. The reciprocal of 6 is $\frac{1}{6} = 6^{-1}$.

But what if you start out with a negative exponent in the denominator? What happens then? Look at the fraction $\frac{1}{3^{-4}}$. If you write the denominator as a fraction, you get $\frac{1}{\frac{1}{3^4}}$. Then, changing the *complex fraction* (a fraction with a fraction in it) to a division problem: $\frac{1}{\frac{1}{3^4}} = 1 \div \frac{1}{3^4} = 1 \cdot \frac{3^4}{1} = 3^4$. So, to simplify a fraction with a negative exponent in the denominator, you can do a switcheroo: $\frac{1}{3^{-4}} = 3^4$.

Here are two examples of simplifying the fractions by getting rid of the negative exponents:

- $\frac{x^2y^3}{3z^{-4}} = \frac{x^2y^3z^4}{3}$
- $\frac{4a^3b^5c^6d}{a^{-1}b^{-2}} = 4a^3a^1b^5b^2c^6d = 4a^4b^7c^6d$

Putting Powers to Work

Because exponents are symbols for repeated multiplication, one way to write $(x^3)^6$ is $x^3 \cdot x^3 \cdot x^3 \cdot x^3 \cdot x^3 \cdot x^3$. Using the multiplication rule, where you just add all the exponents together, you get $x^{3+3+3+3+3+3} = x^{18}$.

To raise a power to a power, use this formula: $(x^n)^m = x^{n \cdot m}$. In other words, when the whole expression, x^n, is raised to the mth power, the new power of x is determined by multiplying n and m together.

Here are a few examples of simplifying using the rule for raising a power to a power:

- $\mathbf{\left(6^{-3}\right)^4 = 6^{(-3)(4)} = 6^{-12} = \frac{1}{6^{12}}}$: You first multiply the exponents; then rewrite the product to create a positive exponent.

- $\mathbf{(x^{-2})^{-3} = x^{(-2)(-3)} = x^6}$.
- $\mathbf{(3x^2y^3)^2 = 3^2x^{(2)(2)}y^{(3)(2)} = 9x^4y^6}$**:** Each factor in the parentheses is raised to the power outside the parentheses.

Circling around Square Roots

When you do square roots, the symbol for that operation is a radical, $\sqrt{\ }$. A cube root has a small 3 in front of the radical; a fourth root has a small 4, and so on.

The radical is a non-binary operation (involving just one number) that asks you, "What number times itself gives you this number under the radical?" Another way of saying this is if $\sqrt{a} = b$, then $b^2 = a$.

Finding square roots is a relatively common operation in algebra, but working with and combining the roots isn't always so clear.

Expressions with radicals can be multiplied or divided as long as the root power *or* the value under the radical is the same. Expressions with radicals cannot be added or subtracted unless *both* the root power *and* the value under the radical are the same.

Here are some examples of simplifying the radical expressions when possible:

- $\mathbf{\sqrt{2} \cdot \sqrt{3} = \sqrt{6}}$**:** These *can* be combined because it's multiplication, and the root power is the same.
- $\mathbf{\sqrt{8} \div \sqrt{4} = \sqrt{2}}$**:** These *can* be combined because it's division, and the root power is the same.
- $\mathbf{\sqrt{2} + \sqrt{3}}$**:** These *cannot* be combined because it's addition, and the value under the radical is not the same.
- $\mathbf{4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3}}$**:** These *can* be combined because the root power and the numbers under the radical are the same.

Here are the rules for adding, subtracting, multiplying, and dividing radical expressions. Assume that *a* and *b* are positive values.

- $\boldsymbol{m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}}$**:** Addition and subtraction can be performed if the root power and the value under the radical are the same.
- $\boldsymbol{\sqrt{a}\sqrt{a} = \sqrt{a^2} = |a|}$**:** The number a can't be negative, so the absolute value insures a positive result.
- $\boldsymbol{\sqrt{a}\sqrt{b} = \sqrt{ab}}$**:** Multiplication and division can be performed if the root powers are the same.
- $\boldsymbol{\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}}$.

When changing from radical form to fractional exponents:

- $\boldsymbol{\sqrt[n]{a} = a^{\frac{1}{n}}}$**:** The *n*th root of *a* can be written as a fractional exponent with *a* raised to the reciprocal of that power.
- $\boldsymbol{\sqrt[n]{a^m} = a^{\frac{m}{n}}}$**:** When the *n*th root of a^m is taken, it's raised to the $\frac{1}{n}$th power. Using the "powers of powers" rule, the *m* and the $\frac{1}{n}$ are multiplied together.

This rule involving changing radicals to fractional exponents allows you to simplify the following expressions.

Here are some examples of simplifying each expression, combining like factors:

- $\boldsymbol{6x^2 \cdot \sqrt[3]{x} = 6x^2 \cdot x^{\frac{1}{3}} = 6x^{2+\frac{1}{3}} = 6x^{\frac{7}{3}}}$.
- $\boldsymbol{3\sqrt{x} \cdot \sqrt[4]{x^3} \cdot x = 3x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} \cdot x^1 = 3x^{\frac{1}{2}+\frac{3}{4}+1} = 3x^{\frac{9}{4}}}$**:** Leave the exponent as $\frac{9}{4}$. Don't write the exponent as a mixed number.
- $\boldsymbol{4\sqrt{x} \cdot \sqrt[3]{a} = 4x^{\frac{1}{2}}a^{\frac{1}{3}}}$**:** The exponents can't really be combined, because the bases are not the same.